JGP - Vol. 5, n. 4, 1988

Virasoro-Gelfand-Fuks type algebras, Riemann surfaces, operator's theory of closed strings

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *In this paper we construct the operator fields of the Riemann surfaces of arbitrary genus. The corresponding operator theory of interacting strings can be considered as the direct development of Virasoro-Mandelstam theory for* $g \geq 0$ *and its unification with Polyakov-Belavin-Knizhnik theory.*

This review sums up the results of three authors' papers [1-3]. The theory of the bosonic string in critical dimension $\mathcal{D} = 26$, which was proposed by Polyakov-Belavin-Knizhnik and others [4-6] is based on the path integral approach. The g-loops contribution to the partition functions, to the string amplitude can be represented in this approach in terms of finite-dimensional integrals over the modular space of Riemann surfaces of the genus g.

On the other hand the «old» bosonic string theory by Virasoro-Mandelstam et al. [7-8] had existed only for the case of «genus $g = 0$». This theory was based on the usual operator's approach to the quantum theory. The creation and annihilation operators for non-interacting bosonic string were introduced using the Fourier expansion for the co-ordinates $X^\mu(\sigma), 0 \leq \sigma \leq 2\pi$. Then in the corresponding Fock space the

Key-Words: Virasoro-Gelfand-Fuks algebras, Riemann surfaces, operator fields, strings.
1980 MSC: 30 F 30.

«physical» states can be defined using the action of the Lie algebra of the symmetry group of this theory – the group of reparametrization of the circle S^1 (more precisely the Virasoro-Gelfand-Fuks algebra which is its central extention). Each of these states generates the so-called Verma-module – the infinite-dimensional highest weight representation playing important role in the whole theory. All those algebraic objects are not clearly presented in the Polyakov et al. theory.

The main purpose of our papers is the construction of the operator fields on the Riemann surfaces of arbitrary genus.

The corresponding operator theory of interacting strings can be considered as the direct development of Virasoro-Mandelstam theory for $g \geq 0$ and its unification with Polyakov-Belavin-Knizhnik theory.

1. TENSORS ON THE RIEMANN SURFACES. ANALOGUES OF THE FOURIER-LAURENT BASES. ALGEBRAIC FUNCTIONS AND VECTOR FIELDS. SPINORS.

Let's consider the simpliest one-string «diagrams» (Riemann surfaces) corresponding to the world sheet of the string which can in the intermediate moments break up into a few components, than glue some of them and etc. The admissible processes can be described in the following way. The triple (Γ, P_+, P_-) would be called a «one-string diagram», where Γ-compact Riemann surface of the genus $g \geq 0$, P_+, P_--two points on it. The holomorphic co-ordinates in the neighbourhoods of these points would be denoted by $z_\pm(Q)$, where $z_\pm(P_\pm) = 0$.

There exists the unique differential dk with the following properties:

a) it has simple poles at $P_\pm$ with the residues ± 1 and is holomorphic on Γ outside $P_\pm$;

b) the function $\operatorname{Re} k(z)$ is single-valued on Γ (i.e. all the periods of the differential dk on $\gamma \backslash (P_+ \cup P_-)$ are purely imaginary).

The function $\operatorname{Re} k(z)$ would be denoted by $\tau(z)$ and would be called «time». We shall denote the curves $\tau(z) = \text{const} = \tau$ by C_τ, the domains $\tau_1 \leq \tau \leq \tau_2$ by $C_{\tau_1 \tau_2}$ (Riemann annulus). The curves C_τ for $\tau \to \pm\infty$ tends to small circles around the points $P_\pm$, respectively. In the neighbourhoods of points $P_\pm$ the canonical co-ordinates $z_\pm$ can be introduced so that the differential dk would have the form:

$$(1.1) \qquad \begin{aligned} dk &= dz_+/z_+ \qquad (\text{near } P_+), \\ dk &= -dz_-/z_- \quad (\text{near } P_-). \end{aligned}$$

For $g = 0$ we have: $\Gamma = CP^1 = S^2$, $z_+ = z$, $z_- = w = z^{-1}$, $P_+ = 0$, $P_- = \infty$. For $g > 0$ the domains of definition of canonical co-ordinates do not intersect.

REMARK. The general «multistring diagrams» can be represented by the set $(\Gamma, P_{+i}, P_{-j}, c'_i, c''_j)$, where P_{+i}, P_{-j} are the points of the surface Γ and $c'_i, -c''_j$ are positive real numbers, such that

$$\sum_{i=1}^{n} c'_i + \sum_{j=1}^{m} c''_j = 0. \tag{1.2}$$

There exists the unique meromorphic differential dk on Γ with simple poles at P_{+i}, P_{-j} and the residues c'_i, c''_j at these points and such that the function $\operatorname{Re} k(z)$ is single-valued on Γ. Again we shall denote this function by $\tau(z)$ and call it «time». The theory of multistring diagrams would be considered in detail in our papers to follow.

Let's consider the tensors of the weight λ on the Riemann surfaces. In local co-ordinates holomorphic tensor of the weight is defined as the value of the form:

$$f = f(z)(dz)^{\lambda}$$

with the following transformation law under the changing of local co-ordinate

$$f(z) \to f(z(w)) \left(\frac{dz}{dw}\right)^{\lambda}$$

The definition of the tensors of the complex weight λ require the introduction of some additional structures on the Riemann surfaces. We shall consider them in some special cases below. The definition and investigation of tensors for the integer λ can be obtained without any difficulties. The important case $\lambda = \frac{1}{2}$ (spinors) requires the introduction of a spinor structure.

The most important cases which are necessary for the construction of the operator theory are as follows:

$\lambda = -1$ (vector fields)
$\lambda = 0$ (scalars)
$\lambda = \frac{1}{2}$ (spinors)
$\lambda = 1$ (differentials)
$\lambda = 2$ (quadratic differentials)

Let's denote the value

$$S = \frac{g}{2} - \lambda(g-1)$$

by $S = S(\lambda, g)$. From the Riemann-Roch theorem it follows that:

LEMMA 1. *For any «one-string» diagram* (Γ, P_+, P_-) *in the general position, any integer* λ, *integer* $n+\frac{g}{2}$ *(except for the cases which would be listed below) there exists the unique, up to the constant factor, tensor* f_n^λ *with the following analytical properties:*

a) *tensor* f^λ *is holomorphic on* Γ *except for the points* $P_\pm$, *where it possibly has the poles of finite orders:*

b) *near the points* $P_\pm$ *it has the form*

$$f_n^\lambda = \text{const} \cdot z_\pm^{\pm n - S(\lambda, g)} (1 + O(z_\pm))(dz_\pm)^\lambda. \tag{1.4}$$

∎

We shall denote by M_λ the space of tensors meromorphic on Γ with the poles only at the points $P_\pm$.

The exceptional cases are as follows:

$$g = 1, \quad n = \frac{1}{2},$$

$$g > 1, \quad \lambda = 0, 1, |n| \leq \frac{g}{2}.$$

In these cases we can define f_n^λ using the following asymptotics near $P_\pm$.

$$\begin{aligned} &\lambda = 0 : f_n^0 = 0(1) z^{n - \frac{g}{2} - 1}, \ f_n^0 = z^{-n - \frac{g}{2}} \cdot 0(1) \\ &\lambda = 1 : f_n^1 = 0(1) z_+^{n + \frac{g}{2}} dz, \ f_n^1 = 0(1) z_-^{-n + \frac{g}{2} - 1} dz_-, |n| \leq \frac{g}{2} \end{aligned} \tag{1.5}$$

The nature of these exceptions is very simple. For $g = 1$ there exists holomorphic non-zero differential dz on Γ corresponding to euclidian co-ordinate z on Γ. That's why any tensor can be globally presented in the form $f^\lambda = f(z)(dz)^\lambda$, where $f(z)$ is a scalar function. In this case there exists no actual difference between the tensors of different weights.

For $\lambda = 1$ and $g \geq 1$ there exist g holomorphic differentials. These differentials together with the differential dk which was introduced above, generate the $(g + 1)$-dimensional space, corresponding to the indices $|n| \leq \frac{g}{2}$. The choice of the basis in this space is non-canonical; below we shall use conditions (1.5). The case $\lambda = 0$ is dual to $\lambda = 1$ and we shall use the conjugate basis.

There exists the natural scalar product of the tensors of the weights λ and $(1 - \lambda)$

$$(f^\lambda, g^{1-\lambda}) = \frac{1}{2\pi i} \oint_{C_\tau} f^\lambda g^{1-\lambda}. \tag{1.6}$$

From the definitions (1.4, 1.5) it follows that, after the appropriate choice of constant factors, the bases f_n^λ and $f_n^{1-\lambda}$ are dual

(1.7) $$(f_n^\lambda, f_{-m}^{1-\lambda}) = \delta_{n,m}.$$

Let's now consider the so-called Baker-Akhiezer function which is widely used in the theory of periodic difference operators with scalar coefficients (in particular, in the theory of periodic Toda lattice, discrete KdV equation [9, 10] and in the theory of general commutative difference operators [11, 12]). This function $F_{\mathcal{D}}(n, z)$ is defined for any triple $(\Gamma, P_\pm)$ and any set of g points $\mathcal{D} = (\gamma_1 + \ldots + \gamma_g)$ in general position. It has the asymptotics near points

$$F_{\mathcal{D}}(n, z) = \text{const} \cdot z_\pm^{\pm n}(1 + O(z_\pm)), \quad n \in Z,$$

and outside them – simple poles at the points $\gamma_1, \ldots, \gamma_g$. If all these poles tend to the points $P_\pm$ then

$$\text{for} \quad g = 2q: \ F_{\mathcal{D}}(n, z) \to f_n^0(z) = F_{\mathcal{D}_0}(n, z), \mathcal{D}_0 = qP_+ + qP_-,$$
$$\text{for} \quad g = 2q+1: \ F_{\mathcal{D}}(n, z) \to f_{n-\frac{1}{2}}^0 = F_{\mathcal{D}_0}(n, z), \mathcal{D}_0 = qP_+ + (q+1)P_-.$$

Hence for $\lambda = 0$ our basis f_n^0 is the particular case of the Baker-Akhiezer function corresponding to the special choice of the set of poles at points $P_\pm$. For $\lambda \neq 0$ the consideration of the values

$$F_{\mathcal{D}_\lambda}(n, z) = f_n^\lambda / f_0^\lambda$$

reduces the general case to scalar Baker-Akhiezer functions. The divisor $\mathcal{D}_\lambda$ of the poles of this function for $\lambda \neq 0, 1$ in general do not contain the points $P_\pm$ and are not special. We shall call tensors $f_n^\lambda(z) = F^\lambda(n, z)$ the «tensor Baker-Akhieser function». It is the common eigenfunction of the commutative difference (in respect to the variable n) operators and the curve Γ is the curve of «spectral parameter».

The general theta-functional formulae for $F_{\mathcal{D}}(n, z)$ were obtained in [11]. Hence the exact formulae for our basic tensors f_n^λ can be easily obtained from the soliton theory. From these formulae it follows that coefficients of the difference operators (the eigenfunctions of which are $f_n^\lambda(z)$) are quasi-periodic functions of the variable n. Hence, one can use the avaraging procedure and obtain the following analogue of the Fourie-Laurent expansion.

THEOREM 1. *Let C_τ be non-singular. For any smooth tensor f^λ of the weight λ on C_τ expansion*

(1.8) $$f^\lambda(\sigma) = \sum_n f_n^\lambda(\sigma) \times \left(\frac{1}{2\pi i} \oint_{C_\tau} f^\lambda(\sigma') f_{-n}^{1-\lambda}(\sigma') d\sigma'\right)$$

is valid. The same expansion is valid for the tensors $f^\lambda(z)$ *which are holomorphic in the Riemann annulus* $C_{\tau_1\tau_2}$. *The convergence of this series is the same as in the ordinary Laurent-Fourier series.* ∎

(The theorem is valid for singular contours C_τ but smoothness conditions in these cases are slightly more rigorous. The theorem is valid independently of whether the C_τ contour is connected or not.)

The important properties of our bases f_n^λ (which immediately follow from the definition) are their almost-graduated structure in respect to the multiplication

$$f_n^\lambda f_m^\mu = \sum_{|k|\le \frac{g}{2}} Q_{n,m}^{\lambda,\mu,k} f_{n+m-k}^{\lambda+\mu}, \tag{1.9}$$

$$[e_n, f_m^\lambda] = \sum_{|k|\le g_0} R_{n,m}^{\lambda,k} f_{n+m-k}^\lambda, g_0 = 3\frac{g}{2}. \tag{1.10}$$

Here and below we use the notations

$$e_n = f_n^{-1}, A_n = f_n^0, \Phi_n = f_n^{\frac{1}{2}}, dw_n = f_{-n}^1, d^2\Omega_n = f_{-n}^2. \tag{1.11}$$

For exceptional cases $\lambda, \mu = 0, 1$; $|n| \le \frac{g}{2}$ or $|m| \le \frac{g}{2}$ the sum in (1.9, 1.10) must include the additional terms with $|k| = \frac{g}{2} + \epsilon$, $|k| = g_0 + \epsilon$ where $\epsilon = 1, 2$ (see exactly in [1, 2]).

DEFINITION. *An almost-graduated* (N*-graduated) algebra* L (or module M over L) is an algebra (or module) which can be expanded into direct sum of the subspaces

$$L = \sum_j L_j, \qquad M = \sum_j M_j$$

so that

$$L_i L_j \in \sum_{|k|\le N} L_{i+j-k}, \qquad L_i M_j \in \sum_{|k|\le N} M_{j+i-k}. \tag{1.12}$$

According to (1.9), we have the commutative almost-graduated algebra A^Γ of scalar functions on Γ with the basis A_n and the $N = g_0$-graduated Lie algebra L^Γ with its basis e_n.

All the spaces M_λ of the tensors are N-graduated modules over algebras $A_\Gamma(N = \frac{g}{2} + 1)$ and $L^\Gamma(N = g_0, g > 1)$.

The other examples of the almost graduated modules in context of the soliton theory are discussed by the authors in their work [1]. Here we shall briefly consider only the simplest generalizations of the modules M_λ which are particularly important in the case $\lambda = \frac{1}{2}$. The corresponding modules $M_\lambda^{\rho,\sigma,p}$ depend on the set of data $(\Gamma, P_\pm, \rho, \sigma, p)$ where: σ is the line which connects the points $P_\pm$; $\rho : \pi_1(\Gamma) \to C^*$ is the character of the fundamental group $\pi_1(\Gamma)$; p is an arbitrary complex number.

The module $M_\lambda^{\rho,\sigma,p}$ is the space of the (multi-valued) tensors of the weight λ which are holomorphic on Γ except for the points $P_\pm$ and the line σ. Along this line σ the boundary values of such tensors must satisfy the following relation

$$f^+(z) = e^{2\pi i p} f^-(z), \quad z \in \sigma.$$

For each closed cycle $\gamma \in \pi_1(\Gamma)$ the changing of $f^\lambda \in M^{\rho,\sigma,p}$ when moving along γ, is the multiplication f^λ by the complex number $\rho(\gamma)$.

The cases $\rho(\gamma) = \pm 1$ correspond to spinor structures on Γ.

LEMMA 2. *If $\lambda = \frac{1}{2}, (\nu - p)$ is a half-integer and ρ is the representation in the «general position», there exists the unique, up to the constant factor, spinor $\Phi_\nu(z, \rho) \in M^{\rho,\sigma,p}$, which in the neighbourhoods of the points $P_\pm$ has the form of*

$$\Phi_\nu(z, \rho) = \text{const} \cdot z_\pm^{\pm\nu - \frac{1}{2}}(1 + O(z_\pm))(dz_\pm)^{\frac{1}{2}} \tag{1.13}$$

For an integer ρ the tensor Φ_ν does not depend on σ. ∎

The representation ρ such that $\rho(\gamma) = \pm 1$ is in general position iff (for general Γ) the corresponding spinor structure is even.

The analogues of the almost-graduated properties (1.9, 1.10) are valid for spinors

$$\Phi_\nu(z, \rho)\Phi_\mu(z, \rho^{-1}) = \sum_{|k| \le \frac{g}{2}} a_{\nu,\mu}^k dw_{k-\nu-\mu}(z), \tag{1.14}$$

$$[e_n, \Phi_\nu] = \sum_{|k| \le g_0} C_{n,\nu}^k \Phi_{n+\nu-k}. \tag{1.15}$$

In case $\rho(\gamma) = \pm 1$ (spinor structures, $\rho = \rho^{-1}$) spinors Φ_ν are square roots from the meromorphic differentials.

We can unite the data $\rho, e^{2\pi i p}$ considering them as the character

$$\hat{\rho} : \pi_1(\Gamma - (P_+ \cup P_-)) \to C^*.$$

2. RIEMANN ANALOGUES OF HEIZENBERG AND VIRASORO ALGEBRAS

The reparametrization group – i.e. the group of the diffeomorphisms of the circle – is a natural group of symmetries of the theory of closed bosonic string. Its Lie algebra is the algebra of the vector fields on the circle. For each parametrization φ of the circle, the subalgebras Z_+, Z_-, Z_0 can be introduced

$$Z = Z_+ + Z_0 + Z_-, \tag{2.1}$$

where Z_0 is one dimensional and generated by the vector field $e_0 = z\frac{\partial}{\partial z}, z = e^{i\varphi}$ and subalgebras $Z_\pm$ are generated by the vector-fields

$$e_n \in Z_+,\ e_{-n} \in Z_-,\ n > 0;\ e_n = z^{n+1}\frac{\partial}{\partial z}.$$

The algebra Z, as it has been shown by Gelfand-Fuks [13], has a single cohomology class - central extension, which is defined by the cocycle

$$\chi(f,g) = \frac{1}{48\pi i}\oint_{S'}(f'''g - g'''f)\,dz, \tag{2.2}$$

where $f = f(z)\frac{\partial}{\partial z}$, $g = g(z)\frac{\partial}{\partial z}$ are vector-fields on the circle. In this extention the commutators of the elements have the form

$$[f,g] = (f'g - g'f)\frac{\partial}{\partial z} + \chi(f,g)\cdot t,\ [f,t] = 0.$$

This extented algebra would be called «Gelfand-Fuks algebras» and denoted by Z_c.

At the beginning of the 70-ies this algebra was independently discovered by physicists (Virasoro, Mandelstam and others) in the context of constructing the operator quantization of the string. To be more precise, they have found the Z-graduated subalgebra L_c of Z_c which consists of the central element t and all trigonometrical polynomial vector-fields (i.e. all finite combinations of vector-fields e_n and t)

$$\begin{gathered} L_c \subset Z_c,\ L_c = \sum_{n\in Z} L_c^n,\ L_c^n = (e_n),\ n\neq 0, L_c^0 = (e_0,t). \\ [e_n,e_m] = (m-n)\,e_{n+m} + t\cdot\frac{n^3-n}{12}\delta_{n+m,0} \end{gathered} \tag{2.4}$$

This Z-graduated algebra L_c is called «Virasoro algebra». It has the obvious decomposition, corresponding to (2.1)

$$L_c = L_+ + L_0 + L_-.$$

The most important representations of the Virasoro algebra are the so-called right «Verma modules» $W^R_{h,c}$. The module $W^R_{h,c}$ has the «highest vector» Ψ^R which satisfies to the following relations

$$L_+\Psi^R = 0,\ e_0\Psi^R = h\Psi^R,\ t\Psi^R = c\Psi^R. \tag{2.5}$$

The space of the representation $W^R_{h,c}$ is the space of the finite sums of the basis vectors of the form:

$$\Psi^R, e_{-n_1} e_{-n_2} \ldots e_{-n_k}\Psi^R, \quad n_1 \geq n_2 \geq \ldots \geq n_k > 0.$$

For the general pairs (h, c) the module $W^R_{h,c}$ is irreducible. The reducible modules $W^R_{h,c}$ correspond to the pairs (h, c) such that $P_{n,m}(e, h) = 0$ where $P_{n,m}$ is one of the «Kac-polynomials».

The left Verma module $W^L_{h,c}$ has the generating vector Ψ^L

$$\Psi^L L_- = 0, \quad \Psi^L e_0 = h\Psi^L, \quad \Psi^L t = c\Psi^L.$$

The details of the representation theory of the Virasoro algebra can be found in [14].

The simple and important example of the reducible Verma module corresponds to the «vacuum sector», where $h = 0$ and c is arbitrary. If $h = 0$ then the non-trivial vector $\tilde{\Psi}^R = e_{-1}\Psi^R$ satisfies the relations:

$$L_+\tilde{\Psi}^R = 0, \quad e_0\tilde{\Psi}^R = -\tilde{\Psi}^R, \quad t\tilde{\Psi}^R = c\tilde{\Psi}^R.$$

If $W^R_{h,c}$ is reducible then the irreducible representation can be obtained as the factor-module of $W^R_{h,c}$ over the ideals which are generated by all the «singular-vectors» $\Psi^R_j \in W^R_{h,c}$ such that $L_+\Psi^R_j = 0$.

The vacuum sector in the string theory is irreducible. Hence, we must have in this sector the relation $e_{-1}\Psi^R_{\text{vac}} = 0$.

Let's consider now the Heizenberg algebra which is more simple then the Virasoro algebra. This algebra has the basis $a_n, a^+_n, t, n > 0$, with the following commutators

$$\begin{aligned} &[a_n, a_m] = [a^+_n, a^+_m] = [t, a_n] = [t, a^+_n] = 0, \\ &[a_n, a^+_m] = n\delta_{n,m} \cdot t. \end{aligned} \tag{2.8}$$

This algebra A also has the decomposition

$$\begin{aligned} A = &A_+ + A_0 + A_- \\ &(a_n)\ (t)\ (a^+_n). \end{aligned} \tag{2.9}$$

The analogues of right Verma modules for this algebra are well-known in the elementary quantum theory. The generating vector Ψ^R_{vac} in this case is called «in-vacuum»

(2.10) $$A_+\Psi^R_{\text{vac}} = 0, \quad t\Psi^R_{\text{vac}} = \Psi^R_{\text{vac}}.$$

The space of the representation of algebra A (Verma modules) is called the right (or in) bosonic Fock space of the scalar theory. Its basic vectors have the form

(2.11) $$a^+_{i_1} \dots a^+_{i_k} \Psi^R_{\text{vac}}.$$

The left (or out-) Fock space can be defined in similar way. As it was shown at the beginning of 70-ies the physicists, the operators

(2.12) $$L_k = \frac{1}{2} \sum_{n=-\infty}^{\infty} : a_n a_{-n+k} : ; a^+_m = a_{-m}.$$

generate Virasoro algebra with central change $c = 1$, $L_k \Leftrightarrow -e_k$. Here the «normal ordering» has been used:

(2.13) $$\begin{aligned} &:a_n a_m := a_n a_m, : a^+_n a^+_m := a^+_n a^+_m, \\ &:a_n a^+_m :=: a^+_m a_n := a^+_m a_n. \end{aligned}$$

The geometrical realization of Verma modules over Virasoro algebra has been proposed by Feigin-Fuks [14]. For any complex p the tensors of the weight λ on the circle with the «multiplicator» p

$$f = f(\varphi)(d\varphi)^\lambda, f(\varphi + 2\pi) = e^{2\pi i p} f(\varphi)$$

can be defined. In the space M^p_λ of such tensors the tensors

$$f^{\lambda,p}_n = z^{n-\lambda+p}(dz)^\lambda$$

are basic.

The finite linear combinations of the basic vectors generate the module M^p_λ. Let's consider the right semi-infinite forms – exterior products of the form

(2.16) $$f^{\lambda,p}_{n_1} \wedge f^{\lambda,p}_{n_2} \wedge \dots \wedge f^{\lambda,p}_{n_s} \wedge \dots$$

such that the consequence $(n_1, n_2, \dots, n_s, \dots)$ becomes stable from some number on. This means that for some k and k_0, $n_s = s + k - 1 > k_0$. The simplest vectors of such form are

(2.17) $$\Psi^R_k = f^{\lambda,p}_k \wedge f^{\lambda,p}_{k+1} \wedge f^{\lambda,p}_{k+2} \wedge \dots$$

The action of the vector fields e_n on the forms (2.16) can be correctly defined with the help of Leibnitz rule for $n \neq 0$. The attempt to define the action of e_0 with the help of the commutator's relations leads to the representation of the central extention of L, i.e. to the representation of the Virasoro algebra (2.4) with the central charge

$$c = -12\lambda^2 + 12\lambda - 2.$$

The space of all right semi-infinite forms is the direct sum

$$W^R_{\lambda,p} = \sum_{k \in Z} W^R_{\lambda,p,k},$$

where $W^R_{\lambda,p,k}$ is generated by Virasoro algebra from vector Ψ^R_k (2.17). The corresponding highest weight equals $h_k = \frac{1}{2}(p + k - \lambda) \times (1 - p - k - \lambda)$.

In the same way the space of the left semi-infinite forms can be defined. The space $W^L_{\lambda,p,k}$ is the space of finite linear combinations of the left semi-infinite form:

$$(2.20) \qquad \ldots \wedge f^{\lambda,p}_{m_s} \wedge \ldots \wedge f^{\lambda,p}_{m_2} \wedge f^{\lambda,p}_{m_1}$$

where $m_s = k - s + 1$ for $s > k_0$ for some k_0.

$$W^L_{\lambda,p} = \sum_{k \in Z} W^L_{\lambda,p,k}, \quad \Psi^L_k = \ldots \wedge f^{\lambda,p}_{k-2} \wedge f^{\lambda,p}_{k-1} \wedge f^{\lambda,p}_k.$$

This construction of the representation of the Virasoro algebra is based on special Fourier basis in the space of tensors on the circle. Our construction of the operator fields on the Riemann surfaces, the definitions and investigations of their vacuum expectation values (Green functions) widely used the analogues of the semi-infinite form. At this moment, especially, the analogues of the Fourier-Laurent bases, which were introduced above, are necessary.

To begin with we shall introduce the analogues of the Heizenberg and Virasoro algebras in the pure geometrical way.

Let $(\Gamma, P_\pm)$ be an arbitrary one-string diagram. The commutative algebra A^Γ of the meromorphic functions on Γ with the poles at the points $P_\pm$ has the basis $A_n = f^0_n$ which was introduced in § 1. The analogue of the Heizenberg algebra is the Lie algebra with the basis a_n, t the commutators of which have the form of

$$(2.22) \qquad [a_n, a_m] = \gamma_{nm} \cdot t, \; [a_n, t] = 0; \; \gamma_{nm} = \frac{1}{2\pi i} \oint_{C_\tau} A_m dA_n.$$

From (1.4) it follows that $\gamma_{nm} = 0, |n + m| > \text{const} = N$,

$$N = g, \; |n|, |m| > \frac{g}{2}; \; N = g + 1, \; |n| > \frac{g}{2}, |m| \leq \frac{g}{2},$$

$$N = g + 2, |n|, |m| \leq \frac{g}{2}.$$

For $g = 0$, $P_+ = 0, P_- = \infty$ we obtain the ordinary Heizenberg algebra.

The definition of the analogue of the Virasoro algebra requires the introduction of the «projective structure» on the surface $\Gamma\backslash(P_+ \cup P_-)$, i.e. the introduction of the systems of local coordinates, such that they connect each other with the help of projective transformations from the group $SL(2, C)/\pm 1$. Let's define the commutators

$$[e_n, e_m] = \sum_{k=-g_0}^{g_0} c_{nm}^k e_{n+m-k} + \chi_{nm} \cdot t, \tag{2.23}$$

$$[e_n, t] = 0,$$

where the coefficients c_{nm}^k in (2.23) are the same as in (1.10). The cocycle $\chi(f, g)$ for the pair of vector-fields

$$f = f(z)\frac{\partial}{\partial z}, \quad g = g(z)\frac{\partial}{\partial z}$$

in the system of the projective co-ordinates has the form

$$\chi(f, g) = \frac{1}{48\pi i}\oint_\gamma (f'''g - g'''f)\,dz, \tag{2.24}$$

where γ is the arbitrary element of the homology group

$$[\gamma] \in H_1(\Gamma\backslash(P_+ \cup P_-); Z).$$

The formula (2.24) is correctly defined because the value $f'''g - g'''f$ is trasformed as the 1-form under the projective transformation of the co-ordinate z. The cocycle χ can be defined in any (non-projective) system of the co-ordinates with the help of the so-called projective connection R on Γ, which is holomorphic on Γ outside the points $P_\pm$. Such connection is defined in any system of co-ordinates as the function $R(z)$, which is transformed in the following way under the transformation of the co-ordinate:

$$R(z) \to R(z(w))\left(\frac{dz}{dw}\right)^2 + \left[\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2\right], z' = \frac{dz}{dw}. \tag{2.25}$$

The difference between the two projective connections is the quadratic differential.

In the arbitrary system of the holomorphic co-ordinates, where $R \neq 0$, the cocycle (2.24) can be presented in the form of:

$$\chi(f, g) = \frac{1}{48\pi i}\oint_\gamma (f'''g - g'''f\chi - 2R(f'g - g'f))\,dz. \tag{2.26}$$

The cohomology class of these cocycles does not depend on the choice of the projective connection (i.e. on the choice of projective structure on Γ) but for our purpose the choice of the cocycle is important.

THEOREM 2. *All almost-graduated central extensions of the algebra L^Γ (i.e. such extensions of the algebra L^Γ that $\chi(e_n, e_m) = f_{nm} = 0$ if $|n+m| >$ const) are defined by the formulae (2.24) or (2.26), where the class of homology $[\gamma]$ is the class of the cycle C_τ, which separates the points P_+ and P_- on Γ.* ∎

CONJECTURE.

$$H^2(L^\Gamma, R) = H_1(\Gamma \backslash (P_+ \cup P_-), R).$$

It must be mentioned that the Riemann analogue of the Heizenberg algebra which was defined above with the help of the formulae (2.22), is also almost-graduated iff the homology class $[\gamma]$ is the class of the cycle C_τ.

Below we shall consider only almost-graduated central extensions of L^Γ and A^Γ because this structures are very important for our futher constructions.

The complex conjugate anti-holomorphic theory can be constructed completely in the same way. The string theory involves (as for the case $g = 0$) both the holomorphic and anti-holomorphic algebras. But we can confine ourselves to the consideration of holomorphic part only because holomorphic and anti-holomorphic algebras commute with each other.

The Riemann analogue of the Virasoro algebra has two filtrations which are generated by the Taylor expansions near the points $P_\pm$, respectively

a) $L_c^\Gamma : \ldots \supset L_{n-1}^+ \supset L_n^+ \supset \ldots$

b) $L_c^\Gamma : \ldots \supset L_m^- \supset L_{m-1}^- \supset \ldots$

The space L_n^+ is generated by the vector-fields e_j where $j \geq g_0 + n$ (i.e. the expansion of e_j near P_+ begins from $z_+^k, k \geq n+1$).

The space L_m^- is generated by the vector-fields $e_j, j \leq m - g_0$.

We have

$$[L_n^\pm, L_m^\pm] \subset L_{n+m}^\pm.$$

The adjoint algebras

$$\hat{L}_c^\Gamma = \sum_n L_n^+/L_{n+1}^+, \hat{L}_c^\Gamma = \sum_n L_n^-/L_{n-1}^-$$

both isomorphic to the ordinary Virasoro algebra.

The similar filtrations have the Riemann analogues of Heizenberg algebra.

The algebra L_c^Γ, A^Γ and spaces of tensor and spinor fields have the decompositions:

$$A^\Gamma = A_+ + A_0 + A_- \qquad (\lambda = 0),$$
$$L_c^\Gamma = L_+ + L_0 + L_- \qquad (\lambda = -1),$$
$$M_\lambda = M_{+,\lambda} + M_{0,\lambda} + M_{-,\lambda} \quad (\lambda = \tfrac{1}{2}, -1, 0, 1, 2),$$

where $A_+, L_+, M_{+,\lambda}$ are generated by the basic tensors (or spinors) f^λ, n, which at the point P_+ have the zero of the order $\geq s(\lambda)$

$$
\begin{array}{lll}
s(\lambda = 0) = 1 & \text{for } A_+ & \text{(scalars)} \\
s(\lambda = -1) = 2 & \text{for } L_+ & \text{(vector-fields)} \\
s(\lambda = \frac{1}{2}) = 0 & \text{for } \lambda = \frac{1}{2} & \text{(scalars)}
\end{array}
$$

The subspaces $A_-, L_-, M_{-,\lambda}$ are defined in the same way.

In the dual spaces $\lambda \to 1 - \lambda$ the similar decompositions are dual by definition according to the scalar product (1.6). For $\lambda = 1, 2$ we have

$$
\begin{array}{ll}
M_1 = M_{+,1} + M_{0,1} + M_{-,1} & (\lambda = 1, \text{1-form}) \\
M_2 = M_{+,2} + M_{0,2} + M_{-,2} & (\lambda = 2, \text{quadratic differentials})
\end{array}
$$

The space $M_{0,1}$ consists of the holomorphic differentials and the differential dk which has simple poles at the points $P_\pm$. The subspace $M_{0,2}$ is the space of the quadratic differentials holomorphic except for the points $P_\pm$ where they have the poles of the orders not greater than 1. The dimensions of the spaces $M_{0,1}, A_0$ equal to $g + 1$. The dimensions of the spaces L_0 and $M_{0,2}$ equal to $3g + 1$.

The subalgebras A_+, A_- are commutative.

Let's define the subspaces

$$\tilde{M}_{\pm,\lambda} \subset M_\lambda$$

as the space of all the tensors of the weight λ which are holomorphic on Γ except for the point P_+ or P_-, respectively. The intersection $\tilde{M}_{+,\lambda} \cap \tilde{M}_{-,\lambda}$ consists of the tensors which are holomorphic on Γ everywhere. The peculiarity of the case $\lambda = \frac{1}{2}$ (for an even spinor structure) is the properties that

$$\tilde{M}_{+,\frac{1}{2}} + \tilde{M}_{-,\frac{1}{2}} = M_{\frac{1}{2}}, \tilde{M}_{+,\frac{1}{2}} \cap \tilde{M}_{\pm,\frac{1}{2}} = \emptyset.$$

In this case

$$\tilde{M}_{\mp,\frac{1}{2}} = M_{\mp,\frac{1}{2}}.$$

The spaces $M_{\pm,\frac{1}{2}}$ are dual to each other with respect to the scalar product (1.6).

The analogues of Verma modules for the algebras A^Γ, L_c^Γ are defined with the help of the generating vectors and the following conditions

a) $A_+ \Psi^R_{\text{vac}} = 0$, (right (in) Fock space)
$L_+ \Psi^R_\alpha = 0, e_{g_0} \Psi^R_\alpha = h \Psi^R_\alpha$ (right Verma module)
$t \Psi^R_\alpha = c \Psi^R_\alpha$.

b) $\Psi^L_{\text{vac}} A_- = 0$, (left (out) Fock space)

$\Psi^L_\alpha = 0, \Psi^L_\alpha e_{-g_0} = h\Psi^L_\alpha$, (left Verma module)
$\Psi^L_\alpha = c\Psi^L_\alpha$.

The operator belonging to $A_+(A_-)$ are annihilation operators of «in-» («out-») states.

From the existence of the filtrations of A^Γ, L^Γ_c which are generated by the Taylor expansions at the points $P_\pm$ it follows that the closure of these spaces are isomorphic to the ordinary free Fock spaces and Verma modules but with different basic states and different Z-graduated structures.

The «free» basic states are defined with the help of the canonical local co-ordinates (1.1). The transformation matrices $U_\pm$ from the bases $z^n_\pm$ to $A_m = f^0_m$ are triangular. The powers $z^n_\pm$ correspond to the creation and annihilation operators of the free (in) and (out) states for $\tau \to \mp\infty$. Hence, formally the matrix

$$S = U_-^{-1}U_+$$

defines the analogue of the Bogolubov transformation from the basis of free in-states to the basis of free out-states. But for $g > 0$ the matrix S is ill-defined because the elements of the product of the infinite matrices U_-^{-1}, U_+ are given by the series which seem to diverge.

The basic elements of the Riemann analogues of the Virasoro algebra can be represented in terms of the generators of the quadratic expressions as in free case $g = 0$ (Sugawara-type construction). Therefore, the Riemann analogues of the Virasoro algebra are acting in the Fock spaces (in- and out-). The different physical states correspond to the different Verma modules with the different highest weights h. In particular, the vacuum sectors in the right and left Fock spaces correspond to $h = 0$ and are generated by the vectors Ψ^R_{vac} and Ψ^L_{vac}, respectively. The central charge c would be equal to the dimension of physical space $c = \mathcal{D}$.

It must be mentioned once again that in vacuum sectors the following relations are valid:

$$e_{g_0}\Psi^R_{\text{vac}} = 0, \quad e_{g_0-1}\Psi^R_{\text{vac}} = 0,$$
$$\Psi^L_{\text{vac}}e_{-g_0} = 0, \quad \Psi^L_{\text{vac}}e_{-g_0+1} = 0.$$

(We shall call them «the regularity conditions of the vacuum».)

3. THE RIEMANN ANALOGOUS OF THE HEIZENBERG AND VIRASORO ALGEBRAS IN THE STRING THEORY

The phase space of the classical $\mathcal{D}$-dimensional string in the Euclidean or Minkovsky spaces is the space of 2π-periodic functions $X^\mu(\sigma)$ and 2π-periodic 1-form $P^\mu(\sigma)$ with Poisson brackets

$$\{p^\nu(\sigma'), X^\mu(\sigma)\} = \eta^{\nu\mu}\Delta(\sigma,\sigma'), \tag{3.1}$$

where $\Delta(\sigma,\sigma')$ is the δ-function on the circle (which is a scalar function of the variable σ and 1-form of the variable σ') i.e.

$$f(\sigma) = \oint f(\sigma')\Delta(\sigma,\sigma')d\sigma'. \tag{3.2}$$

This definition is adeguate only in the case of free string because in this case there are no topological bifurcations of the string.

For a one-string diagram $(\Gamma, P_\pm)$ the contour C_τ plays the role of the string position at the fixed moment τ.

Let $X^\mu(Q), P^\mu(Q)$ be an operator-valued scalars and 1-forms for $Q \subset \Gamma$ which commute with each other at different moments of «time» τ. Naive quantization of (3.1) gives us

$$[X^\mu(Q), P^\nu(Q')] = -i\eta^{\mu\nu}\Delta_\tau(Q,Q'), \tag{3.3}$$

where Δ_τ is the δ-function on the contour C_τ. As it follows from the results of § 1, this δ-function can be represented in the form

$$\Delta_\tau(Q,Q') = \frac{1}{2\pi i}\sum_n A_n(Q)dw_n(Q'), Q, Q' \subset C_\tau. \tag{3.4}$$

Let's expand X^μ and P^μ in our analogue of the Fourier series

$$\begin{aligned} X^\mu(Q) &= \sum X^\mu_n A_n(Q), \\ P^\mu(Q) &= \sum P^\mu_n dw_n(Q). \end{aligned} \tag{3.5}$$

The direct consequence of (3.3), (3.4) is the commutator relations

$$[P^\mu_n, X^\mu_m] = \frac{1}{2\pi}\eta^{\mu\nu}\delta_{m,n}. \tag{3.7}$$

The coefficients of the expansion of the 1-form $dA_m = \sum_n \gamma_{mn}dw_n$ are given by the formulae (2.22). Then the operators α^μ_n which are defined from the expansion

$$(\pi P^\mu + \partial_\sigma X^\mu)d\sigma = \sum_n \alpha^\mu_n dw_n \tag{3.8}$$

are equal to

$$\alpha^\mu_n = \pi P^\mu_n + \sum_m \gamma_{mn}X^\mu_m. \tag{3.9}$$

LEMMA 3. *The operators α_n^μ satisfy the commutator relations of the Heizenberg algebra*

$$[\alpha_n^\mu, \alpha_m^\nu] = \gamma_{nm}\eta^{\mu\nu}. \tag{3.10}$$ ■

The complex conjugate differentials, scalars, tensors lead to the definition of the operators $\bar{\alpha}_n^\mu$ which commute with α_n^μ :

$$X^\mu(Q) = \sum_n \bar{X}_n^\mu \bar{A}_n(Q), P^\mu(Q) = \sum_n \bar{P}_n^\mu d\bar{\omega}_n(Q),$$

$$\bar{J}^\mu(Q) = (\partial_\sigma X^\mu - \pi P^\mu)\, d\sigma = \sum \bar{\alpha}_n^\mu \overline{d\omega}_n(Q),$$

$$[\bar{\alpha}_n^\mu, \bar{\alpha}_m^\nu] = \eta^{\mu\nu}\bar{\gamma}_{nm}, \quad [\alpha_n^\mu, \bar{\alpha}_m^\nu] = 0.$$

The Fock spaces of in- and out-states can be defined using the vacuum vectors $\Psi_{\text{vac}}^R, \Psi_{\text{vac}}^L$ and the conditions

$$\begin{aligned} \alpha_n^\mu \Psi_{\text{vac}}^R &= \bar{\alpha}_n^\mu \Psi_{\text{vac}}^R = 0, \quad n > \tfrac{g}{2}, n = -\tfrac{g}{2}. \\ \Psi_{\text{vac}}^L \alpha_n &= \Psi_{\text{vac}}^L \bar{\alpha}_n = 0, \quad n \le -\tfrac{g}{2}. \end{aligned} \tag{3.11}$$

In the classical case the densities of the Hamiltonian and momentum are the linear combinations of the values

$$T = \frac{1}{2}(X_\sigma + \pi P)^2 = \frac{1}{2}\mathcal{I}^2, \bar{T} = \frac{1}{2}(X_\sigma - \pi P)^2 = \frac{1}{2}\bar{\mathcal{I}}^2.$$

The definitions of the corresponding quantum operators require, as usual, the definition of the«normal ordering». Let's dissect the integer (or half-integer) plane of pairs (n, m) into two parts $\sum^\pm$ such that $\sum^+$ differs from the integer half-plane $m \le n$ only in the finite number of points. The definition of the normal ordering depend on the choice of $\sum$

$$:\alpha_n\alpha_m: = \begin{cases} \alpha_n\alpha_m, & (n,m) \in \sum^-, \\ \alpha_m\alpha_n, & (n,m) \in \sum^+ \end{cases} \tag{3.12}$$

In [2] a bit more general definitions of normal ordering were considered.

We can define the quantum operators

$$\begin{aligned} T(Q) &= \frac{1}{2} : I^2 := \frac{1}{2}\sum : \alpha_n\alpha_m : d\omega_n(Q) d\omega_m(Q), \\ \bar{T}(Q) &= \frac{1}{2} : \bar{I}^2 := \frac{1}{2}\sum : \bar{\alpha}_n\bar{\alpha}_m : \overline{d\omega}_n(Q)\overline{d\omega}_m(Q). \end{aligned} \tag{3.13}$$

They are quadratic differentials on C_τ. Hence, they can be expanded in the series

$$T = \sum L_k d^2\Omega_k(Q), \quad \bar{T} = \sum \bar{L}_k d^2\overline{\Omega_k}(Q). \tag{3.14}$$

The quadratic expressions of L_k through α_n^μ follow from (3.13).

DEFINITION. *The normal ordering is admissible if the conditions of the regularity of vacuum*

$$L_{g_0}\Psi^R_{\text{vac}} = L_{g_0-1}\Psi^R_{\text{vac}} = 0;\ \Psi^L_{\text{vac}}L_{-g_0} = \Psi^L_{\text{vac}}L_{-g_0} = 0$$

are fulfilled.

THEOREM 3. *The operators $e_k = -L_k$, where L_k are given by the formulae (3.13), (3.14) satisfy the commutator relations (2.23) of the Riemann analogues of the Virasoro algebra with the central charge $t = \mathcal{D}$. The cocycle χ_{nm} depends on the choice of the normal ordering but his cohomology class does not depend on this choice. For the admissible normal ordering the corresponding projective connection is holomorhphic on Γ (for other normal ordering it has the poles at points $P_\pm$).* ∎

The proof of the theorem is given in [2, 3], where the examples of the admissible normal ordering are discussed.

Let's consider the geometrical realization of the modules over the Riemann analogues of the Heizenberg and Virasoro algebras which are similar to (2.16-2.21). The bases of spaces of the right and left semi-infinite forms can be defined by the same formulae (2.16-2.21) using the bases f^λ_n. We shall denote these spaces by $W^R_\lambda = \sum_k W^R_{\lambda,k}, W^L_\lambda = \sum_k W^L_{\lambda,k}$ omitting in this denotation the dependence of these spaces on the diagram $(\Gamma, P_\pm)$ and the number ρ.

The almost-graduated structure of all modules under consideration provides the simple proof of the following lemma.

LEMMA 4. *The action of the vector-fields e_n on the space of semi-infinite forms can be correctly defined with the help of the Leibnitz rule (and also the action e_n on f^λ_m) for $|n| > g_0$. This action can be extended to the representation of our Virasoro-type algebra with the central charge depending on the tensor weight λ*

$$t = c = -12\lambda^2 + 12\lambda - 2.$$

The highest weight of the restriction of this representation on the subspace $W^R_{\lambda,k}(W^L_{\lambda,k})$ generated by vector $\Psi^R_k(\Psi^L_k)$ is given by the formulae (4.9) in [1]. ∎

For our purposes the most important cases are as follows:

$$\lambda = \frac{1}{2},\ c = 1 \quad \text{(for the physical scalar bosonic fields)},$$

$$\lambda = -1, 2,\ c = -26 \quad \text{(for the ghost fields)}.$$

Now we shall consider in detail the first case $\lambda = \frac{1}{2}$. Let's consider the following normalization of the basic Fourier-Laurent-type tensors

$$f_n^\lambda(z) = \begin{cases} z_+^{n-S}(1+O(z_+))(dz_+)^\lambda, \\ \varphi_{n,\lambda}^- z_-^{-n-S}(1+O(z_-))(dz_-)^\lambda, \quad S = S(\lambda, g) \end{cases} \tag{3.15}$$

We shall call it - «in-normalization». It is unique up to the transformation $z_+ \to qz_+, f_n^\lambda \to q^{n-S} f_n^\lambda$. It is natural to call the normalization (3.16) – «out-normalization».

$$\tilde{f}_n^\lambda(z) = (\varphi_{n,\lambda}^-)^{-1} f_n^\lambda(z) = \begin{cases} \varphi_{n,\lambda}^+ z_+^{n-S} & (1+O(z_+))(dz_+)^\lambda \\ z_-^{-n-S} & (1+O(z_-))(dz_-)^\lambda \end{cases} \tag{3.16}$$

In particular, for $\lambda = \frac{1}{2}$ we define $f_n^{\frac{1}{2}} = \Phi_n, \tilde{f}^{\frac{1}{2}} = \tilde{\Phi}_n$, where n is half integer.

The generating vectors $\Psi_k^R(\Psi_{k'}^L)$ for any λ in the geometrical realization have the form

$$\begin{aligned} \Psi_k^R &= (f_k^\lambda \wedge f_{k+1}^\lambda \wedge \ldots) = \Lambda_{j\geq k} f_j^\lambda, \\ \Psi_{k'}^L &= (\ldots \wedge f_{k'-1}^\lambda \wedge f_{k'}^\lambda) = \Lambda_{j\leq k'} f_j^\lambda. \end{aligned} \tag{3.17}$$

Below we shall use only «in-normalization».

Regularity conditions of the vacuum.

The generating vectors of the form (3.17) can play the role of the in- (out-) vacuum vector only if the numbers k, k' is such that the corresponding semi-infinite form coincides (up to the constant factor) with the exterior product of *all* the positive powers of the local parameter

$$\begin{aligned} |0_\lambda> = \Psi_k^R &= (1 \wedge z_+ \wedge z_+^2 \wedge \ldots) \\ <0_\lambda| = (\ldots \wedge z_-^2 \wedge z_- \wedge 1) &= \Psi_{k'}^L \Big(\prod\nolimits_{n\leq k'} \varphi_{n,\lambda}^-\Big)^{-1} \end{aligned} \tag{3.18}$$

(in the latest equality at this moment we consider the product $\prod_{n\leq k'} \varphi_{n,\lambda}^-$ as formal). From the asymptotics (1.4), it follows that

$$k = k(\lambda) = S(\lambda, g) = \frac{g}{2} - \lambda(g-1), k' = k'(\lambda) = -S(\lambda, g) \tag{3.19}$$

LEMMA 5. *The vacuum vectors (3.18) satisfy the regularity conditions (2.29).* ∎

Later we shall discuss the physical motivation of the regularity conditions (which were given by A. Polyakov and others in the different language). Later we shall consider also the problem of the regularization of the product $\prod_{n\leq k} \varphi^{-}_{n,\lambda}$ which formally is diverging.

There exists the natural scalar product between the spaces of the right and left semi-infinite forms. For the basic forms $f \in W^L_\lambda, g \in W^R_\lambda$ let's consider the product $f \wedge g$. If this infinite (in both directions) form coincides after the permutation with the standard form (the exterior product of all basic tensors f^λ_n) then we define

$$\langle f, g\rangle = (-1)^\epsilon, \qquad f \wedge g = (-1)^\epsilon \bigwedge_{j=-\infty}^{\infty} z^j_+, \tag{3.20}$$

where ϵ is the sign of the corresponding permutation. In other cases the product $\langle f, g\rangle = 0$ would be equal to zero. The scalar products of any elements $f \in W^L_\lambda, g \in W^R_\lambda$ can be defined by the linearity. The basic forms are the exterior products of the tensors f^λ_n in the «in-normalization».

LEMMA 6. *If k and k' are given by the formulae (3.19) (i.e. the corresponding generating vectors (3.17) can play the roles of the vacuum vectors), their scalar product $\langle \Psi^L_k, \Psi^R_k\rangle \neq 0$ iff $\lambda = \frac{1}{2}, k = \frac{1}{2}, k' = -\frac{1}{2}$.* ∎

A few words should be said about the normalization of the vacuum vectors. It is natural to define the vector $|0\rangle = \Psi^R_{\text{vac}}$ using «in-normalization», and vector $\langle 0| = \Psi^L_{\text{vac}}$-using «out-normalization». The normalized out-vacuum differs from the vector $\Psi^L_{-\frac{1}{2}}$ (which is taken in the «in-normalization») up to factor

$$\langle 0| = \left(\prod_{n<0} \varphi^{-}_{\frac{1}{2},n}\right)^{-1} \Psi^L_{-\frac{1}{2}} = \bigwedge_{n<0} \tilde{\Phi}_n. \tag{3.21}$$

By the definition which was introduced above, we have $\langle \Psi^L_{-\frac{1}{2}}, \Psi^R_{\frac{1}{2}}\rangle = 1$ Therefore,

$$\langle 0_{\frac{1}{2}} | 0_{\frac{1}{2}}\rangle = \left(\prod_{n<0} \varphi^{-}_{n,\frac{1}{2}}\right)^{-1}. \tag{3.22}$$

(We still consider the product in right-hand side of this equality formally. It was shown by Iengo (private communication) that after the regularization it coincides with the determinant of Dirac operator).

LEMMA 7. *The operators L_n acting in the spaces of right and left semi-infinite forms are self-adjoint in respect to the scalar product $\langle , \rangle$ which was defined above, i.e.*

$$\langle f, L_n g\rangle = \langle f L_n, g\rangle, \quad f \in W^L_\lambda, g \in W^R_\lambda \tag{3.23}$$

Below the normalized in- and out- vacuums would be denoted by $|0\rangle$ $\langle 0|$. ∎

Let's consider the spaces $\mathcal{H}^{\pm}$ of the Dirac fermions. They are generated by the fermion operators ψ_ν, ψ_ν^+ with the half-integer indices which satisfy the commutator relations

$$[\psi_\nu, \psi_\nu]_+ = [\psi_\mu^+, \psi_\nu^+] = 0, \qquad [\psi_\mu, \psi_n^+] = \delta_{\mu+\nu,0}. \tag{3.24}$$

The «vacuums» $|0_F\rangle, \langle 0_F|$ are defined by the relations

$$\begin{aligned} &\psi_\nu|0_F\rangle = \psi_\nu^+|0_F\rangle = 0, \nu > 0, \\ &\langle 0_F|\psi_\mu = \langle 0_F|\psi_\mu^+ = 0, \mu < 0. \end{aligned} \tag{3.25}$$

If we suppose that the operators ψ_ν, ψ_ν^+ have the «charge» 1 or -1, respectively, then

$$\mathcal{H}^{\pm} = \sum_{p=-\infty}^{\infty} \mathcal{H}_p^{\pm}, \qquad p\text{-charge}. \tag{3.26}$$

The natural isomorphism between $\mathcal{H}^{\pm}$ and $W_{\frac{1}{2}}^R, W_{\frac{1}{2}}^L$ can be obtained, if we consider the correspondences

$$\begin{aligned} &\psi_\nu \to \phi_\nu \wedge && \text{(multiplication on } \phi_\nu), \\ &\psi_\nu^+ \to \tfrac{\partial}{\partial \phi_\nu} && \text{(differentiation)} \qquad , \\ &\langle 0_F| \to \left(\prod\nolimits_{\nu<0} \varphi^-_{\frac{1}{2},\nu}\right) < 0|, && |0_F\rangle \to |0\rangle. \end{aligned} \tag{3.27}$$

Let $\Phi_\nu^+ = \Phi_\nu(z, \rho^{-1})$ be, by definition, the dual half-differential (spinor). According to (1.13)

$$\frac{1}{2\pi i} \oint_{C_\tau} \Phi_\nu \Phi_\nu^+ = \delta_{\nu+\mu,0}. \tag{3.28}$$

Let's introduce the «fermionic operator-fields»

$$\begin{aligned} &\psi(z,\rho) = \sum_\nu \psi_\nu \Phi_{-\nu}(z,\rho), \\ &\psi^+(z,\rho) = \sum_\nu \psi_\nu^+ \Phi_{-\nu}^+(w,\rho), \phi_\nu^+(z,\rho) = \phi_\nu(z,\rho^{-1}). \end{aligned} \tag{3.29}$$

THEOREM 4. *The chronological ordering product of the operators* $\psi(z,\rho)\psi^+(w,\rho)$, *where* $\tau(z) > \tau(w)$ *is defined correctly. For* $z \to w$ *the operator expansion*

$$\psi(z,\rho)\psi^+(w,\rho) = \frac{\sqrt{dz\,dw}}{z-w} + \mathcal{I}(z,\rho) + O(z-w) \tag{3.30}$$

is fullfilled. The coefficients of the expansions

$$\mathcal{I}(z,\rho) = \sum \alpha_n(\rho)\,dw_n(z) \tag{3.31}$$

satisfy the commutator relations of the generalized Heizenberg algebra (3.32)

$$\left[\alpha_n(\rho), \alpha_m(\rho)\right] = \gamma_{nm}\,. \tag{3.32}$$

■

Let's define the analogue $S_p(z,w,\rho)$ of the Söger kernel. They are uniquely determined by the following analytical properties. With respect to the variables z, w they are the multi-valued tensors of the weight $\frac{1}{2}$ (half-differentials) which are transformed along the contours on Γ according to the representations ρ and ρ^{-1}, respectively. For the fixed w the kernel S_p is

a) holomorphic on Γ except for the points $P_\pm$ and $z = w$ and the contour σ connecting the points $P_\pm$;

b) for the boundary values of S_p of contour σ the relation

$$S_p^+ = e^{2\pi i p} S_p^-$$

is valid.

c) in the neighbourhoods of the points $P_\pm$ it has the form

$$S_p = z_\pm^{\pm p} O(1)(dz_\pm)^{\frac{1}{2}}.$$

The analytical properties of S_p in respect to w (for the fixed z) are the same as for z after the interchanging

$$p \longrightarrow -p, \quad z \longrightarrow w.$$

Near the diagonal $z = w$ the kernel S_p has the form

$$S_p(z,w,\rho) = \frac{\sqrt{dz dw}}{z-w} + ds_p(z,\rho) + O(z-w)\,. \tag{3.33}$$

LEMMA 8. a) *the expansion*

$$S_p(z,w,\rho) = \sum_{\nu \le p - \frac{1}{2}} \Phi_\nu(z,\rho)\Phi^+_{-\nu}(w,\rho) \tag{3.34}$$

is valid for $\tau(z) > \tau(w)$; *b) the expansion*

$$S_p(z,w,\rho) = -\sum_{\nu\geq p+\frac{1}{2}} \Phi_\nu(z,\rho)\Phi^+_{-\nu}(w,\rho) \tag{3.35}$$

is valid for $\tau(z) < \tau(w)$. *(Here* $\nu - \frac{1}{2} - p \in Z$ *in both cases.)* ∎

The main term $ds_p(z,p)$ of the regular part of the expansion S_p near the diagonal $z = w$ is 1-differential, which is holomorphic on Γ except for the points $P_\pm$ where it has simple poles with the residues $\pm p$. For $p = 0$ this differential is holomorphic on Γ.

Using the result of lemma for $p = 0$ we obtain the equality

$$\mathcal{I}(z,\rho) = \sum_{\nu,\mu} :\psi_\nu\psi^+_\mu: \Phi_{-\nu}(z,\rho)\Phi^+_{-\mu}(w,\rho) + ds_0(z,\rho). \tag{3.36}$$

The comparison of (3.36) and (3.31) gives

$$\alpha_n(\rho) = \sum_{\nu,\mu} a^n_{\nu,\mu}(\rho) :\psi_\nu\psi^+_\mu: + a_n(\rho), \tag{3.37}$$

where

$$a^n_{\nu,\mu} = \frac{1}{2\pi i}\oint_{C_\tau} A_n\Phi_{-\nu}\Phi^+_{-\mu}, \quad a_n = \frac{1}{2\pi i}\oint_{C_\tau} A_n ds_0, \tag{3.38}$$

$$a^n_{\nu,\mu} = \begin{cases} |n-\nu-\mu| > \frac{g}{2}, & |n| > \frac{g}{2} \\ |n-\nu-\mu| > \frac{g}{2}+1, & |n| \leq \frac{g}{2}. \end{cases} \tag{3.39}$$

The differential ds_0 is holomorphic on Γ. That's why

$$a_n = a_n(\rho) = 0, \quad |n| > \frac{g}{2}, \quad n = -\frac{g}{2}. \tag{3.40}$$

(recall, that $A_{-\frac{g}{2}} = 1$ according to our choice of the basis 1,3).

DEFINITION. *The vacuum expectation value of the operator* H *is given by the formula*

$$\langle H\rangle_\rho = \frac{\langle 0_\rho|H|0_\rho\rangle}{\langle 0_\rho|0_\rho\rangle}.$$

The vacuum expectation value of the operators equals to ([3])

(3.41) $$\langle \psi(z,\rho)\psi^{+}(w,\rho)\rangle_{\rho} = S_0(z,w,\rho).$$

In the modem physical literature the formula (3.41) is playing the role of the definition of the propagator of the fermionic fields without any construction of the fields proper.

The following equality comes from (3.37)

(3.42) $$\langle \alpha_n(\rho)\rangle_{\rho} = a_n(\rho).$$

For even spinor structure we have from $\rho = \rho^{-1}$ that

$$S_0(z,w,\rho) = -S_0(w,z,\rho)$$

Therefore, $ds_0(z,\rho) = 0, \rho = \rho^{-1}$.

LEMMA 9. *For an even spinor structures*

$$\langle \alpha_n(\rho)\rangle_{\rho} = 0$$

for all integer (or half-integer) n. ∎

The usual Vick Theorem is valid for the chronological product of the fermionic fields on any Riemann surface

(3.43) $$\Psi(z_1)\ldots\Psi(z_n) = \sum_I \pm \prod_{i,j\in I} \langle \Psi(z_i)\Psi(z_j)\rangle : \prod_{k\notin I} \Psi(z_k) :,$$

where $\Psi(z)$ are the fields $\psi(z,\rho)$ or $\psi^{+}(z,\rho)$. Here in addition to (3.41) we have

$$\langle \psi(z,\rho)\psi(w,\rho)\rangle = 0, \ \langle \psi^{+}(z,\rho)\psi^{+}(w,\rho)\rangle = 0.$$

4. THE ENERGY-MOMENTUM PSEUDO-TENSOR AND OPERATORS EXPANSIONS

As it was mentioned in § 3, the energy-momentum tensor depends on the choice of the normal ordering. Now we shall introduce the invariant value – the energy momentum pseudo-tensor.

Let's consider the chronological product of the current operators.

THEOREM 5. *The chronological product $\mathcal{I}^\mu(z)\mathcal{I}^\mu(w)$, where $\tau(z) > \tau(w)$, is correctly defined. For $z \to w$ the following expansion is valid*

$$\sum_\mu \mathcal{I}^\mu(z)\mathcal{I}^\mu(w) = \mathcal{D}\frac{dzdw}{(z-w)^2} + 2\tilde{T}(z) + O(z-w). \tag{4.1}$$

For any projective connection R, which is holomorphic on Γ outside the punctures $P_\pm$, the coefficients of the expansion of the operator-valued quadratic differential

$$T(z) = \tilde{T}(z) - \frac{\mathcal{D}}{2}R(z) = \sum_k L_k d^2\Omega_k(z)$$

have the commutator relations of the Virasoro-type algebra (2.23, 2.26), corresponding to the projective connection R. ∎

The proof is given in [3].

DEFINITION. *The value $\tilde{T}(z)$ is called an energy-momentum pseudo-tensor.*

THEOREM 6. *The chronological product $\tilde{T}(z)\tilde{T}(w), \tau(z) > \tau(w)$, is correctly defined. For $z \to w$ we have*

$$\tilde{T}(z)\tilde{T}(w) = \frac{\mathcal{D}}{2(z-w)^4} + \frac{2\tilde{T}(z)}{(z-w)^2} + \frac{\tilde{T}_z(z)}{z-w} + O(1). \tag{4.2}$$ ∎

The vacuum expectation values of the products $\mathcal{I}(z_i)\tilde{T}(z_j), \ldots,$ can be easily obtained from these definitions and Vick theorem. For example,

$$\langle \mathcal{I}^\mu(z)\mathcal{I}^\mu(w)\rangle = \lim_{z_1\to z}\lim_{w_1\to w}\left\langle\left(\psi(z)\psi^+(z_1) - \frac{1}{z-z_1}\right)\times\right.$$
$$\left.\times\left(\psi(w)\psi^+(w_1) - \frac{1}{w-w_1}\right)\right\rangle.$$

Using the Vick theorem we obtain that

$$\langle \mathcal{I}^\mu(z)\mathcal{I}^\mu(w)\rangle_\rho = -S_0(z,w,\rho)S_0(w,z,\rho). \tag{4.3}$$

The expansion

$$-S_0(z,w,\rho)S_0(w,z,\rho) = \frac{dzdw}{(z-w)^2} + R_\rho(z) + O(z-w) \tag{4.4}$$

defines the so-called Segue-type projective connection $R_\rho(z)$. It must be specially mentioned that this projective connection does not depend on the punctures $P_\pm$. The comparison of the formulae (4.3), (4.4) and the definition of $\tilde{T}(z)$ gives

$$\langle \tilde{T}(z)\rangle_\rho = \frac{\mathcal{D}}{2} R_\rho(z). \tag{4.5}$$

EXAMPLE. Let $g = 1$. We have 3 even half-periods $\omega_\alpha, \alpha = 1, 2, 3$, which correspond to 3 spinor structures. The Segue kernel have the form

$$S_0(z, w, \rho) = \frac{\sigma(z - w + \omega_\alpha)}{\sigma(z - w)\sigma(\omega_\alpha)} e^{-\eta_\alpha(z-w)}$$

Finally, we obtain that

$$\langle \tilde{T}(z)\rangle_\alpha = \frac{\mathcal{D}}{2} \mathcal{P}(\omega_\alpha),$$

where $\sigma, \mathcal{P}$-elliptic Weiershtrass functions. This result coincides with the result of [16] where it was obtained from the different approach, using the Ward identities.

5. A FEW REMARKS ABOUT «GHOST SECTOR»

The Polyakov-Faddeev-Popov ghost-fields in the string theory have the tensor weights -1 and 2 and are fermionic. We shall define them by

$$b(z) = \sum_n b_n d^2\Omega_n(z), \; c(z) = \sum_n c_n e_n(z).$$

The coefficients C_n, c_m have the ordinary commutators

$$[b_n, b_m]_+ = [c_n, c_m]_+ = 0, \; [b_n, c_m]_+ = \delta_{n,m} \tag{5.1}$$

As it was shown in [17], the definitions of the energy-momentum of the ghost-field and the operator of the BRST-charge can be easily generalized for the case of the Riemann surfaces of the genus $g > 0$ with the help of the bases which were introduced in § 1.

We do not consider in detail these definitions but shall make a few remarks.

The full Fock space includes the tensor product of the «physical and «ghost» sectors. In particular, the vacuum vector has to be the tensor product of the «physical» and «ghost» vacuum vectors. The regularity conditions of the ghost vacuum vector have (according to the results of the §, 3 for $\lambda = 2$) the form

$$|0_2\rangle = \Psi_k^R, \quad k = k(2) = -g_0 + 2,$$
$$\langle 0_2| = \Psi_k^L, \quad k' = -k(2).$$

(5.2) $$b_n|0_2\rangle = 0, \quad n \geq g_0 - 1, \qquad c_n|0_2\rangle = 0, \quad n < g_0 - 1,$$

(5.3) $$\langle 0_2|b_n = 0, \quad n \leq -g_0 + 1, \quad \langle 0_2|c_n = 0, \quad n > -g_0 + 1.$$

The operators b_n can be represented in the spaces of the semi-infinite forms W_2^R, W_2^L as the operators of the exterior multiplications on f_n^2 and the operators c_n can be represented as the derivation with respect to f_n^2.

As it was emphasized in § 3 the scalar product of the ghost-vacuum must be equal to zero

(5.4) $$\langle 0_2|0_2\rangle = 0.$$

The non-zero expressions can be obtained only in the presence of the insertions. For example

(5.5) $$\begin{aligned} &\langle 0_2|c_{-1}c_0c_1|0_2\rangle = 1, \quad g = 0, \\ &\langle 0_2|b_{-\frac{1}{2}}c_{\frac{1}{2}}|0_2\rangle, \quad g = 1, \\ &\langle 0_2|b_{-g_0+2}\cdots b_{g_0-2}|0_2\rangle = \left(\prod_{n=-\infty}^{-g_0+1} \varphi_{n,2}^{-}\right)^{-1} \neq 0, \quad g > 1. \end{aligned}$$

The operators b_n for $|n| \leq g_0 - 2, g > 1$, correspond to the holomorphic quadratic differentials which are the basis of the cotangent bundle over the modular space of the surfaces of the genus g. That's why the square of the modulus of the value (5.6) defines the measure on the modular space. The connection of this measure with Polyakov-Belavin-Knizhnik measure will be considered in our paper to follow.

One of the main questions is the definitions of the ghost vacuum expectation value for arbitrary operators. The same arguments as for (5.4) show that

$$\langle 0_2|T^{b,c}|0_2\rangle = 0.$$

The value

$$\left\langle 0_2|T^{b,c}\prod_{n=-g_0+2}^{g_0-2} b_n|0_2\right\rangle \neq 0.$$

is non-zero, but there exists ambiguity in such definition of the ghost-vacuum expectation value of $T^{b,c}$, because we can arrange the operators $T^{b,c}$ and b_n in a different way.

6. REMARKS ABOUT THE STATES WITH NON-ZERO MOMENTUM

Here we shall consider only the characters ρ, which correspond to the even spinor structures, and the integer values of momentum.

Let's define the states with the momentum p for a one-component bosonic field. Consider the space of the spinors with the character ρ and the multiplicator $\exp(2\pi i p)$ along the line connecting the points $P_\pm$. This space was denoted by $M^{\rho,\sigma,p}$ and the spaces of the semi-infinite forms, which had been built using their Fourier-type basis, were denoted by $W^R_{\frac{1}{2},p,\sigma,\rho}$, $W^L_{\frac{1}{2},p,\sigma,\rho}$. In the case of integer ρ the dependence of these spaces on the line σ is absent.

Below it will be shown that the states with integer momentum p which we denote by $|p\rangle, \langle p|$ are equal to

$$|p\rangle = \Psi^R_{p+\frac{1}{2}} = \bigwedge_{\nu>p+\frac{1}{2}} \Phi_\nu, \tag{6.1}$$

$$\langle p| = \left(\prod_{\nu\le p-\frac{1}{2}} \varphi^-_{\frac{1}{2},\nu}\right)^{-1} \Psi^L_{p-\frac{1}{2}} \tag{6.2}$$

Here the generating vectors Ψ^R_k, Ψ^L_k belonging to the spaces $W^R_{\frac{1}{2}}, W^L_{\frac{1}{2}}$ were defined in (2.17).

The motivation of the factor in (6.2) is the same as in case of the definition $\langle 0|$. It follows from the difference between the «in-» and «out-» normalization. The correspondance

$$\begin{aligned} &|0\rangle \to |p\rangle, \\ &\langle 0| \to \langle p| \end{aligned} \tag{6.3}$$

and the structure of the modules over L^Γ allow us to plot the so-called «vertex operator»

$$W^R_{\frac{1}{2}} \to W^R_{\frac{1}{2},p,\sigma,\rho},\; W^L_{\frac{1}{2}} \to W^L_{\frac{1}{2},p,\sigma,\rho}.$$

In [3] the following properties

$$\begin{aligned} &\alpha_n|p\rangle = 0, \quad n > \tfrac{g}{2}, \qquad \alpha_{-\frac{g}{2}}|p\rangle = p|p\rangle, \\ &\langle p|\alpha_n = 0, \quad n < -\tfrac{g}{2}, \quad \langle p|\alpha_{-\frac{g}{2}} = p\langle p| \end{aligned} \tag{6.4}$$

have been proved. These equalities are the consequence of the representation of the current operator $\mathcal{I}(z,\rho)$ in the form

$$\mathcal{I}(z,\rho) = \sum_{\nu,\mu} :\psi_\nu \psi^+_\mu :_p \Phi_{-\nu}(z,\rho)\Phi^+_{-\mu}(z,\rho) + ds_p(z,\rho), \tag{6.5}$$

where $:\ :_p$ -the normal ordering in respect to the vector $|p\rangle$. The vectors $|p\rangle$ and $\langle p|$ are annihilated by the operator for $n > g_0$ and $n < -g_0$, respectively, and are the eigenvectors for the operators L_{g_0} and L_{-g_0}

(6.6) $$L_{g_0}|p\rangle = \frac{p^2}{2}|p\rangle, \langle p|L_{-g_0} = \varphi^-_{1,-g_0}\cdot\frac{p^2}{2}\langle p|.$$

The equalities (6.6) mean the following

LEMMA 10. *The equality*

$$\mathcal{A}(\pm\rho, P_+, P_-) = \frac{\langle p|p\rangle_p}{\langle 0_\rho|0_\rho\rangle} = \prod_{\nu=\mp\frac{1}{2}}^{\mp p\pm\frac{1}{2}} (\varphi^-_{\frac{1}{2}}, \nu)^{\pm 1}, p > 0$$

takes place. The value $\mathcal{A}(\pm p, P_+, P_-)$ depends on P_+, P_- as the tensor of the weight $p^2/2$. ∎

REMARK. If we redefine the vertex operator as the operator corresponding to the shift of the indices in out-normalization and redefine respectively the normalization of the vectors we obtain that

$$\mathcal{A}(p, P_+, P_-) \to \mathcal{A}^{-1}(p, P_+, P_-)$$

and the tensor weight of the amplitued in respect to the variables p_+ will be equal to $-\frac{p^2}{2}$.

From the results of the soliton theory the formula

(6.7) $$\varphi^-_{\frac{1}{2},\nu} = \frac{\theta[\rho]\left(\left(\nu - \frac{1}{2}\right)(A(P_+) - A(P_-))\right)}{\theta[\rho]\left(\left(\nu + \frac{1}{2}\right)(A(P_+) - A(P_-))\right)} E^{-2\nu}(P_+, P_-)$$

follow, where $\theta[\rho]$-theta-function with even characteristic; $E(P_+, P_-)$-Prym-form

$$E^2(P_+, P_-) = \frac{\theta^2[m](A(P_+) - A(P_-))}{\left(\sum_i \omega_i(P_+)\theta_i[m]\right)\left(\sum_i \omega_i(P_-)\theta_i[m]\right)}.$$

Let's choose the local co-ordinates $z_\pm$ so that $E^2(P_+, P_-) = 1$. Then the natural regularization gives

(6.8) $$\prod_{\nu\le\frac{1}{2}} (\varphi^-_{\frac{1}{2},\nu})^{-1} = \langle 0_\rho|0_\rho\rangle = \theta[\rho](0).$$

For the integer p we obtain

$$A(p, P_+, P_-) = \frac{\theta[\rho](p(A(P_+) - A(P_-)))}{\theta[\rho](0)} \tag{6.9}$$

Here and above the value $A(P)$ is the Abelian map from the devisor to the Jacobian torus of the curve.

This is the situation for a one component bosonic field and the integer p. In the D-dimensional space we have

$$\frac{\langle \vec{p} | \vec{p}}{\langle \vec{0} | \vec{0} \rangle} = \prod_{\mu=0}^{D-1} A(p^\mu, P_+, P_-) \tag{6.10}$$

(in the Euclidean metric $\eta_{\mu\nu} = \eta_\mu \delta_{\mu\nu}, \eta_\mu = 1$).

The question how to define the amplitude in the case of the Minkovsky space is yet not absolutely clear for the authors. There are two possibilities which were briefly considered in [3].

In the first version the definition of $A(p, P_+, P_-)$ goes through a few steps. The amplitude must be defined first for rational and then for all the real values of momentum. And finally one can extend the result on the imaginary values after the transformation $p \to ip^0$. For the realization of this program it is necessary to introduce into the consideration the contour σ between the points P_+ and P_-. There are different possibilities at this point as well. One can consider all possible contours σ or non-intersecting «time-like» contours. In any case it is necessary to take the avarage value with respect to the choice of σ. In the case of all possible contours σ we obtain

$$\tilde{A}(p, P_+, P_-) = \left(\frac{\theta[\rho](A(P_+) - A(P_-))}{E(P_+, P_-)\theta[\theta](0)} \right)^{p^2} \tag{6.11}$$

This formula can be extended over all complex (for integer p we shall choose the formula (6.11) instead of (6.9)). Therefore, in the Minkovsky case the amplitude has the form

$$\frac{\langle \vec{p} | \vec{p} \rangle}{\langle 0 | 0 \rangle} = \prod_{\mu=0}^{D-1} (\tilde{A}(p^\mu, P_+, P_-))^{\eta_\mu}, \eta_0 = -1, \eta_1 = \ldots = 1, \tag{6.12}$$

where $\tilde{A}$ is given by the formula (6.11).

The second version of the definition of $A(\vec{p}, P_+, P_-)$, which was briefly discussed in [3] is based on other definition of the vertex operator (or normalized state $\langle p|$). If such operator corresponds to the shift of indices of basic form $\tilde{\Phi}_\nu$ in out-normalization for zero component then we obtain even in the case of integer η^μ the formula

$$\frac{\langle \vec{p} | \vec{p} \rangle}{\langle 0 | 0 \rangle} = \prod_{\mu=0}^{D-1} (A(p^\mu, P_+, P_-))^{\eta_\mu}.$$

We hope that our future studies will show which of the two versions is correct.

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Manuscript received: November 9, 1988